5. Let $f(x)$ be a real-valued function defined on the positive reals such that

(i) $f(x) < f(y)$ if $x < y$, and

(ii) $f\left(\frac{2xy}{x+y}\right) = \frac{f(x)+f(y)}{2}$ for all x.

Show that $f(x) < 0$ for some value of x.

Solution by Michel Bataille, Rouen, France.

As f is an increasing function on $(0,\infty)$, either $\lim\limits_{x\to 0+} f(x) = -\infty$ or $\lim\limits_{x\to 0+} f(x) = a$ for some real number a. Assume that the latter holds. In the relation

$$f\left(\frac{2xy}{x+y}\right) = \frac{f(x)+f(y)}{2},$$

fix $x > 0$ and let y approach 0^+. Since $\lim\limits_{y\to 0+} \frac{2xy}{x+y} = 0$, it follows that

$$a = \frac{f(x)+a}{2},$$

hence, $f(x) = a$. Consequently, f would be a constant function, contrary to (i). Thus, $\lim\limits_{x\to 0+} f(x) = -\infty$ and certainly $f(x) < 0$ for some positive x.

Next we turn to solutions of problems of the Second and Third Selection Tests of the 2004 Republic of Moldova, given at [2007: 411–412].

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the relation

$$f(x^3) - f(y^3) = (x^2+xy+y^2)(f(x)-f(y))$$

for all real numbers x and y.

Solved by Michel Bataille, Rouen, France; and Salem Malikić, student, Sarajevo College, Sarajevo, Bosnia and Herzegovina. We give the solution of Malikić.

Taking $y = 0$ in the identity yields $f(x^3) - f(0) = x^2(f(x) - f(0))$. Setting $g(x) = f(x) - f(0)$ we have $g(x^3) = x^2 g(x)$. The following are then equivalent

$$\begin{aligned}
f(x^3) - f(y^3) &= (x^2+xy+y^2)(f(x)-f(y)),\\
g(x^3) - g(y^3) &= (x^2+xy+y^2)(g(x)-g(y)),\\
x^2 g(x) - y^2 g(y) &= x^2 g(x) + xy\,g(x) + y^2 g(x)\\
&\quad - x^2 g(y) - xy\,g(y) - y^2 g(y),\\
0 &= xy\,g(x) + y^2 g(x) - x^2 g(y) - xy\,g(y)\\
0 &= (x+y)(y\,g(x) - x\,g(y)). \qquad (1)
\end{aligned}$$

Taking $y = 1$ in equation (1), we must have $(x+1)\big(g(x) - x \cdot g(1)\big) = 0$. Thus, for all $x \in \mathbb{R}\backslash\{-1\}$, we must have $g(x) = xg(1)$, or equivalently $f(x) - f(0) = x\big(f(1) - f(0)\big)$. This means that $f(x) = kx + c$ for all $x \in \mathbb{R}\backslash\{-1\}$, where $k = f(1) - f(0)$ and $c = f(0)$.

By what we have just done, $f(2^3) = f(8) = 8k + c$ and $f(2) = 2k + c$, thus, taking $x = 2$ and $y = -1$ in the identity for f yields

$$8k + c - f(-1) \;=\; 3\big(2k + c - f(-1)\big)\,.$$

Solving for $f(-1)$ we obtain $f(-1) = k(-1) + c$. Finally, we conclude that $f(x) = kx + c$, where k and c are constants.

Conversely, if $f(x) = kx + c$ where k and c are arbitrary constants, then one readily checks that this f satisfies the required identity for all reals x and y.

7. Let ABC be an acute-angled triangle with orthocentre H and circumcentre O. The inscribed and circumscribed circles have radii r and R, respectively. If P is an arbitrary point of the segment $[OH]$, prove that $6r \le PA + PB + PC \le 3R$.

Solution by Arkady Alt, San Jose, CA, USA.

Let $\overrightarrow{PO} = t\overrightarrow{HO}$, $t \in [0,1]$ and let $X \in \{A, B, C\}$. Then

$$\begin{aligned}\overrightarrow{PX} &= \overrightarrow{PO} + \overrightarrow{OX} \;=\; t\,\overrightarrow{HO} + \overrightarrow{OX} \\ &= t\left(\overrightarrow{HX} + \overrightarrow{XO}\right) \;+\; \overrightarrow{OX} \;=\; (1-t)\,\overrightarrow{OX} \;+\; t\,\overrightarrow{HX}\,.\end{aligned}$$

Since $|\overrightarrow{PX}| = |(1-t)\overrightarrow{OX} + t\overrightarrow{HX}| \le (1-t)\,|\overrightarrow{OX}| + t\,|\overrightarrow{HX}|$, we have

$$\begin{aligned}PA + PB + PC &= \sum_{\text{cyclic}} |\overrightarrow{PA}| \;\le\; \sum_{\text{cyclic}} \Big((1-t)\,|\overrightarrow{OA}| + t\,|\overrightarrow{HA}|\Big) \\ &= 3(1-t)R + t\sum_{\text{cyclic}} HA\,.\end{aligned}$$

For any vertex X, $HX = 2R\cos X$. Also, $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$ and Euler's Inequality, $R \ge 2r$, holds. Thus,

$$\begin{aligned}PA + PB + PC &\le 3(1-t)R \;+\; 2Rt(\cos A + \cos B + \cos C) \\ &= 3(1-t)R \;+\; t(2R + 2r) \\ &\le 3(1-t)R \;+\; t(2R + R) \;=\; 3R\,.\end{aligned}$$

Next we prove the inequality $6r \le PA + PB + PC$ for *any* interior point P in the acute-angled triangle ABC.

For each vertex X let R_X be the distance from P to X. Let h_a, h_b, and h_c be the heights of the triangle to the corresponding side, and let d_a, d_b, and d_c be the distances from P to the corresponding side.

Since $R_A + d_a \geq h_a$ we have $\sum_{\text{cyclic}} (R_a + d_a) \geq \sum_{\text{cyclic}} h_a$. By the Erdös–Mordell Inequality in the form $\sum_{\text{cyclic}} d_a \leq \frac{1}{2} \sum_{\text{cyclic}} R_A$ and the preceding inequality we have $\frac{3}{2} \sum_{\text{cyclic}} R_A \geq \sum_{\text{cyclic}} h_a$, or equivalently $\frac{2}{3} \sum_{\text{cyclic}} h_a \leq \sum_{\text{cyclic}} R_A$.

Since

$$h_a + h_b + h_c \;=\; 2F\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \;\geq\; 2F\left(\frac{9}{a+b+c}\right) \;=\; \frac{9F}{2s} \;=\; 9r\,,$$

where F is the area of triangle ABC, we finally obtain

$$6r \;\leq\; \frac{2}{3}(h_a + h_b + h_c) \;\leq\; R_a + R_b + R_c\,.$$

Equality occurs if and only if P is the circumcenter and $a = b = c$.

9. For all positive real numbers a, b, and c, prove the inequality

$$\left|\frac{4(a^3-b^3)}{a+b} + \frac{4(b^3-c^3)}{b+c} + \frac{4(c^3-a^3)}{c+a}\right| \;\leq\; (a-b)^2 + (b-c)^2 + (c-a)^2\,.$$

Solved by Arkady Alt, San Jose, CA, USA.

Let $G(a,b,c) = (a-b)^2 + (b-c)^2 + (c-a)^2$ and

$$F(a,b,c) \;=\; \frac{4(a^3-b^3)}{a+b} \;+\; \frac{4(b^3-c^3)}{b+c} \;+\; \frac{4(c^3-a^3)}{c+a}\,.$$

It suffices to prove that $F(a,b,c) \leq G(a,b,c)$ for all positive real numbers a, b, and c. Indeed, if $F(a,b,c) < 0$ then under this assumption we have

$$|F(a,b,c)| \;=\; -F(a,b,c) \;=\; F(b,a,c) \;\leq\; G(b,a,c) \;=\; G(a,b,c)\,.$$

For positive a and b we have $\frac{4b^2}{a+b} \geq 3b - a$, since this is equivalent to $4b^2 \geq 3b^2 - a^2 + 2ab$, and hence to $(a-b)^2 \geq 0$. We now have

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{4(a^3-b^3)}{a+b} &= 4\sum_{\text{cyclic}} \frac{a^3+b^3}{a+b} \;-\; 2\sum_{\text{cyclic}} \frac{4b^3}{a+b} \\
&\leq 4\sum_{\text{cyclic}} (a^2 - ab + b^2) \;-\; 2\sum_{\text{cyclic}} b(3b-a) \\
&= \sum_{\text{cyclic}} (4a^2 - 4ab + 4b^2 - 6b^2 + 2ab) \\
&= \sum_{\text{cyclic}} (4a^2 - 2ab - 2b^2) \;=\; \sum_{\text{cyclic}} (a-b)^2\,.
\end{aligned}$$

10. Determine all the polynomials $P(X)$ with real coefficients which satisfy the relation

$$(x^3+3x^2+3x+2)P(x-1) \;=\; (x^3-3x^2+3x-2)P(x)$$

for every real number x.

Solved by Arkady Alt, San Jose, CA, USA. Comment by Michel Bataille, Rouen, France.

This problem was one of the problems of the 2003 Vietnamese Mathematical Olympiad. A solution appeared in this journal at [2007: 90–91].

11. Let ABC be an isosceles triangle with $AC = BC$, and let I be its incentre. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The straight lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the straight lines DF and GE intersect on the circumcircle of the triangle ABC.

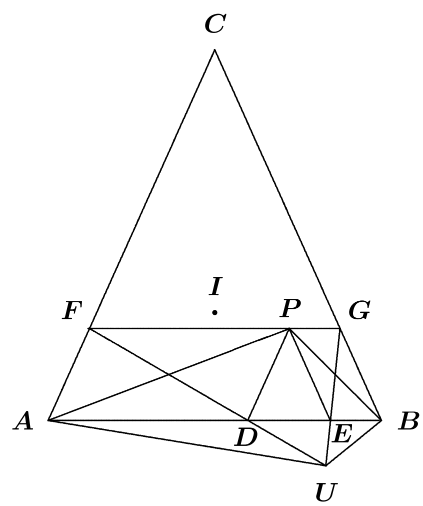
Solved by Ricardo Barroso Campos, University of Seville, Seville, Spain.

For convenience let

$$\angle CAB = \alpha\,,\ \angle ACB = \gamma\,,$$
$$\angle DFP = \xi\,,\ \angle GPB = \omega\,.$$

We then have

$$\begin{aligned}\angle APB &= \angle AIB = \alpha+\gamma\,,\\ \angle APF &= 180^\circ-(\gamma+\alpha)-\omega\\ &= \alpha-\omega\,,\\ \angle GBP &= \alpha-\omega\,,\\ \angle PAF &= \omega\,.\end{aligned}$$



Thus, $GBEP \sim FPDA$, so that $\angle EGP = \angle DFA = 180^\circ-\alpha-\xi = \alpha+\gamma-\xi$.

Let U be the intersection of the lines EG and FD. We then have

$$\begin{aligned}\angle GUF &= 180 - \angle DFP - \angle EGP\\ &= (2\alpha+\gamma) - \xi - (\alpha+\gamma-\xi) = \alpha\,.\end{aligned}$$

Now, $UBGD$ is inscribable, since $\angle DUG = \angle DBG = \alpha$. Also, $PGBD$ is inscribable, since $\angle DPG+\angle GBD = 180^\circ$. Thus, $PGBUD$ is inscribable and $\angle GUB = \angle GPB = \omega$. Similarly, $\angle FUA = \angle FPA = \alpha-\omega$ and $\angle AUB = 2\alpha$; hence, U is on the circumcircle of triangle ABC.

That completes this number of the *Corner*. Send solutions soon!